Vocabulary: *Shapes of Algebra*

<table>
<tr><th>Concept</th><th>Example</th></tr>
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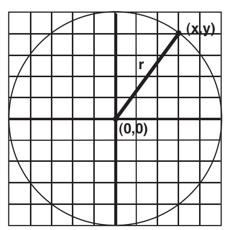
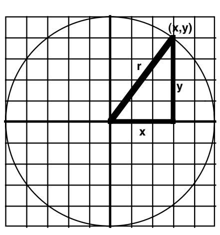
Circle: a shape made of points that are all the same distance from a fixed center point. This distance is the **radius**. See *Covering and Surrounding*.

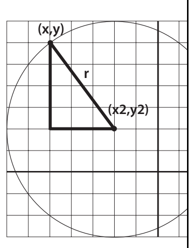
Circle on a coordinate grid: as above. The center may be the origin or some other point.

Figure A



Note: by connecting any point on a circle drawn on a coordinate grid (except for points placed horizontally or vertically from the center point) to the center of the circle, one has drawn the hypotenuse of a right angled triangle. The Pythagorean Theorem (See *Looking for Pythagoras*) gives a relationship between the sides of this triangle. This theorem is the key to writing an equation that describes the relationship between the x and y coordinates of any point on a circle.





Equations of Circles:

- Centered at the origin (0, 0),

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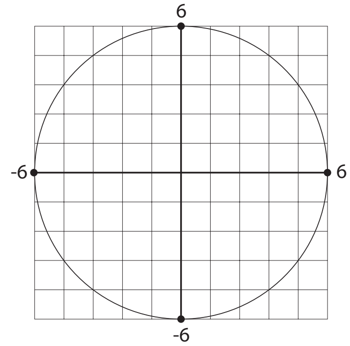
1. Find 4 points on this circle.



The easiest points to see are:
(6, 0), (0, 6), (-6, 0), (0, -6).
For other points we can choose any value for x we wish, say x = 1, and substitute this into the equation for the circle, to find the corresponding value for y. For this we need the equation: see example 2 below.

2. Find the **equation** *of the circle shown above and find 4 points that lie on this circle and satisfy this equation.*

The circle is centered on the origin with radius 6. (We can see the length of the radius most clearly from (0,0) to (6, 0) or any other point where the circle intersects the axes.)
Thus the **equation is**
$x^2 + y^2 = 6^2$.

Substituting any value for x, say x = 1, we find a corresponding value for y:
$1^2 + y^2 = 6^2$, so
$1 + y^2 = 36$, so
$y^2 = 35$, so
y = 5.9 (approx)
Thus (1, 5.9) lies on the circle. Taking advantage of the symmetry of the circle we can see that (1, -5.9) and (-1, 5.9) and (-1, -5.9) will also lie on the circle.

3. Find ***the equation of this circle (not centered on origin)***

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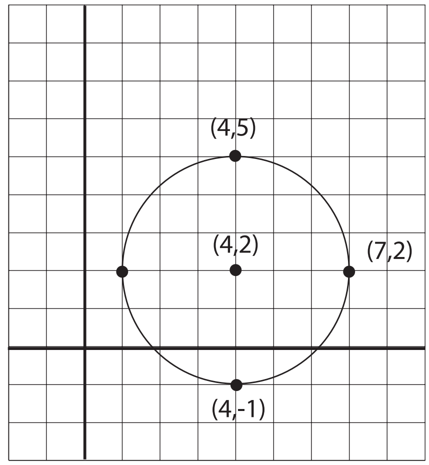
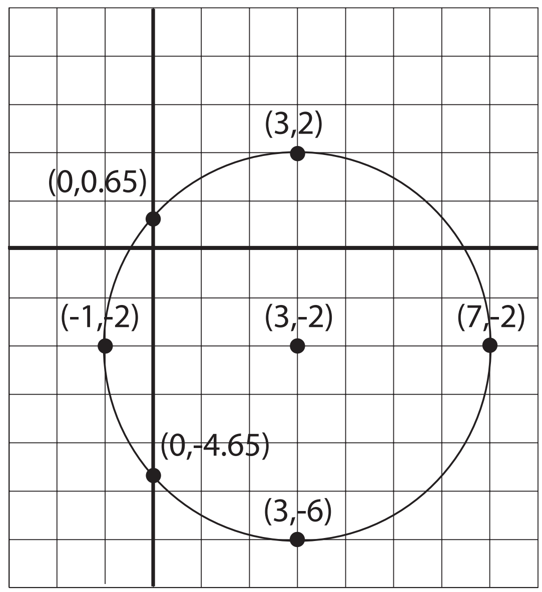
as shown above, $x^2 + y^2 = r^2$. • Centered at some other point (x_2, y_2), radius r, as shown above, $(x - x_2)^2 + (y - y_2)^2 = r^2$. Note: the **Distance Formula** is just a variation on this last equation. If we wish to find the distance between any 2 points (x_1, y_1) and (x_2, y_2) on a coordinate grid we can create a right triangle and apply the Pythagorean theorem to find the square of the distance between the 2 points. Taking the square root give the distance between the points. $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$	 The **center** of the circle is (4, 2) and the **radius** is 3, the distance between (4, 2) and (7, 2). Thus the **equation** is: $(x - 4)^2 + (y - 2)^2 = 3^2$. *4. What is the* ***center*** *of the circle with* ***equation*** *$(x - 3)^2 + (y + 2)^2 = 16$? Find 4 points on the circle.* Comparing this equation with $(x - x_2)^2 + (y - y_2)^2 = r^2$, we see that $(x - 3)^2 + (y - -2)^2 = 4^2$, so (3, -2) is the center of the circle and the radius is 4. To find points on the circle we can substitute any value for x (or y) into the equation to find the corresponding value for y (or x). The circle is not centered on the origin, so it will be more difficult, but not impossible, to take advantage of symmetry to generate more points on the circle. (A drawing would help us use symmetry.) • Say x = 0, then $(0 - 3)^2 + (y + 2)^2 = 16$, so $9 + (y + 2)^2 = 16$, so $(y + 2)^2 = 7$, so, taking square roots, y + 2 = 2.65 (approx.) Thus y = 0.65. This leads to the point (0, 0.65). When we took the square root of both sides there is actually another solution, y + 2 = -2.65, so y = -4.65. This leads to the point (0, -4.65). • Say x = 3, then $(3 - 3)^2 + (y + 2)^2 = 16$, so $(y + 2)^2 = 16$, so, taking square roots, y + 2 = 4, or y = 2, leading to the point (3, 2). Or y + 2 = -4, or

$y = -6$,
leading to the point (3, -6).

- Say $y = -2$, then
  $(x - 3)^2 + (-2 + 2)^2 = 16$, so
  $(x - 3)^2 + 0 = 16$, so
  $x - 3 = 4$, or
  $x = 7$,
  leading to the point (7, -2).
  Or, $x - 3 = -4$,
  leading to the point (-1, --2).

These points are all pictured below. You can see how the points labeled are symmetrically placed.



Linear Equations or Inequalities: are equations (or inequalities) in one or more variables, where any variable terms have power 1. For example, 2x + 7.5 = 11 is a linear equation in 1 variable. Y = 2x + 7.5 is a linear equation in 2 variables.

Solving Linear Inequalities in One Variable Symbolically

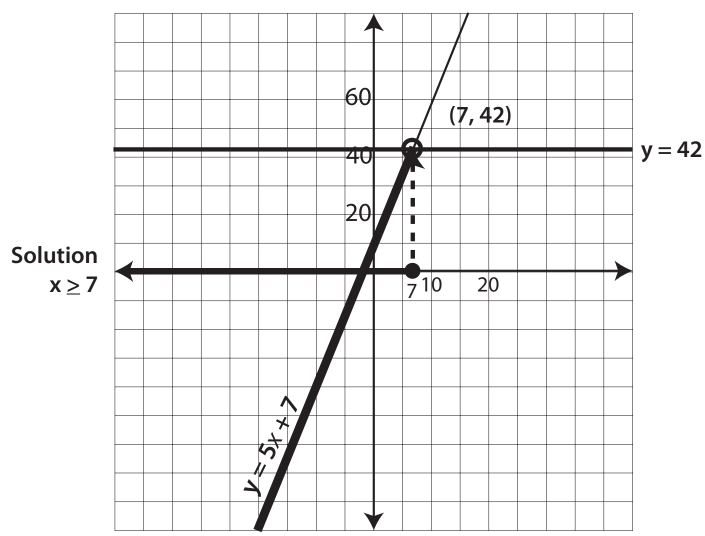
Students already understand how to apply the *Properties of Equality* (See *Moving Straight Ahead*, or *Thinking With Mathematical Models* or *Say It With Symbols*) to solve linear equations in 1 variable. They now have to amend this strategy to solve linear inequalities in 1 variable. Specifically, they find that:

- Adding (or subtracting) the same quantity to each side of a linear inequality will produce an **equivalent** inequality, that is, an inequality with the same solutions, and
- Multiplying or dividing each side of a linear inequality by the same positive quantity will produce an **equivalent** inequality, but
- Multiplying or dividing each side of a linear inequality by a negative quantity will NOT produce an equivalent inequality, unless the direction of the inequality sign is reversed. For example, 7 < 10 is true but multiplying by -1 produces -7 < -10 which is NOT true; this is because if *a* is to the left of *b* on a number line, then –*a* is to the right of –*b* on a number line. We can adjust for this change in order by changing the direction of the inequality sign. Thus,

*5. Solve 5x + 7 ≤ 42 **graphically**.*

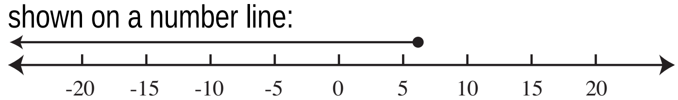
We graph each side of the inequality as a function, i.e., *y* = 5*x* + 7 and *y* = 42. Note that the intersection of these two lines satisfies the equation 5*x* + 7 = 42 and occurs when *x* = 7. (The point (7, 42) lies on both lines.)

However, we need to find when 5*x* + 7 ≤ 42, not just 5x + 7 = 42. The points on the line *y* = 5*x* + 7 satisfying the condition that the *y*-coordinate is less than 42 are all the points on the bolded part of the line shown below. Since the original inequality is asking just for the ***x*-values** of these points, the solutions are the *x*-values corresponding to the points on this bolded section of the line
y = 5*x* + 7. That is, the *x*-values for these points satisfy x < 7. This solution is illustrated on the *x*-axis in the following diagram.



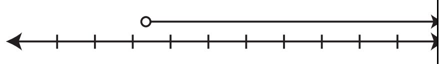
6. Solve 5x + 7 ≤ 42 symbolically.

5x + 7 ≤ 42, so
5x ≤ 35 (subtracting 7 from both sides) so
x ≤ 7.
Note: usually the solution for an inequality in 1 variable is shown on a number line:



The closed circle indicates that x = 7 is part of the solution set.

-2x + 7.5 < 11 is **equivalent** to -2x < 3.5, which, by dividing by -2, is **equivalent** to x > -1.75.
Notice the change of order in the last step. There are an infinite number of values of x which are solutions; these **solutions are usually shown on a number line**:



The open circle on the graph above indicates that x = -1.75 is NOT a part of the solution set.

Note: Prior to this unit students have solved linear inequalities in 1 variable using tabular and graphing methods. (See *Moving Straight Ahead*. For example, they may solve -2x + 7.5 < 11 by thinking first of y = -2x + 7.5, and of the point on the graph (or in the table) that corresponds to the solution of -2x + 7.5 = 11, (that is the point (-1.75, 11) and then checking points on either side of that point to find solutions (for x) where y < 11, not y = 11.) Also see Example 5 in column to the right.

Solving a Linear Inequality in Two Variables Graphically

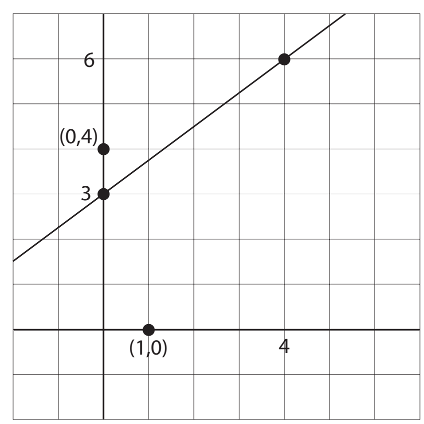
When the linear inequality to be solved has only one variable, for example -2x + 7.5 < 11, then students have a choice of graphical, tabular or symbolic solution strategies (see above and Example 5 in the column to the right) to find the **values of x** which are solutions. But when the linear inequality has 2 variables, for example y < -2x + 7.5, then we are being asked to find **pairs of**

Note: this strategy is very efficient. Unlike the graphing strategy above we do not bring in an extra variable "y" just for the purpose of graphing.

7. Solve 3x – 4y < -12, graphically.

The inequality in two variables is in **standard form.** Although the question asks us to solve graphically we can start by doing some symbol manipulation to get this into **slope intercept form**, which is easy to graph.
3x – 4y < -12, so, subtracting 3x from both sides,
-4y < -12 – 3x, so, dividing both sides by -4,
y > (-12/-4) – (3/-4)x, so
y > 3 + 0.75x. Notice reversal of order.

At each stage in this solution process we produced another inequality that is equivalent to the original. We now have an inequality in **slope intercept** form; that is, the **boundary of the region** of points which are solutions is a line y = 3 + 0.75x, with slope 0.75 and intercept 3. We graph this line and then check which region has solution points for the inequality.



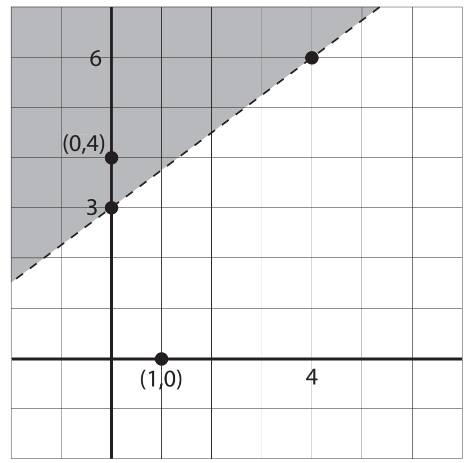
Choosing a point above the line we might select (0, 4). (ANY point clearly above the boundary will do.) Checking this point into y > 3 + 0.75x (or into the original 3x – 4y < -12) we have 4 > 3 + (0.75)(0), which is TRUE. That is, (0, 4) is a solution. It is in the region where all solutions for the inequality are located. We shade this region to show where all solutions are located.

Suppose we had chosen a point below the boundary line, say (1, 0). Then substituting this into the inequality we have 0 > 3 + 0.75(1), which is NOT TRUE. This tells us

solutions (x, y) which make the inequality true. As before, there will be an infinite number of solutions, but this time they can not be shown on a number line, nor stated symbolically (as in x > -1.75). This time the ONLY solution strategy is to graph **the boundary condition**, y = -2x + 7.5, a line, and then locate the points (x, y) that make y < -2x + 7.5 true, by **shading the half plane** that contains the solutions.

Note: should the inequality be in **standard form**, ax + by < c, then it will have to be rewritten in **slope-intercept** format to facilitate graphing. See example 7 to the right.

have 0 > 3 + 0.75(1), which is NOT TRUE. This tells us that (1, 0) is NOT a solution, so is NOT in the region where all solutions to the inequality are located.



The **shaded region** of the graph indicates all solutions of the inequality y > 3 + 0.75x. Notice that the boundary is shown as a **broken line.** This indicates that points on the line
y = 3 + 0.75x are not included in the solution set.

Note: there are symbolic ways to solve linear equations and inequalities in 1 variable. There are only graphical ways to solve linear inequalities in 2 variables.

Systems of Linear Equations in 2 (or more) Variables are 2 (or more) linear equations which are paired together to indicate that a common solution is sought. That is, each linear equation in 2 variables has an infinite number of solutions which satisfy the equation (points which lie on the line representing the equation), but a system made of 2 of these linear equations has a solution only if there is a solution (or many solutions) that makes both of the equations true simultaneously (a point that lies on both lines). Usually a system of linear equations represents 2 contextual conditions that must be satisfied.
The system may have the format:
y = ax + b
y = cx + d (both equations in **slope-intercept** format, see *Moving Straight Ahead*),
Or
ax + by = c
dx + ey = f (both equations in **standard format**),
Or some combination of these.

Solving Systems of Linear Equations in 2 variables can be done graphically or symbolically.

- **Graphic Solution of Systems —**The graphic method involves producing straight-line graphs for each equation and then reading coordinates of intersection points as the solution(s). Since this method relies on pictorial representation of the equations, coordinates of the intersection point can only be estimated and may not even appear in the graphing window chosen for display of

*8. Solve this system **graphically**.*

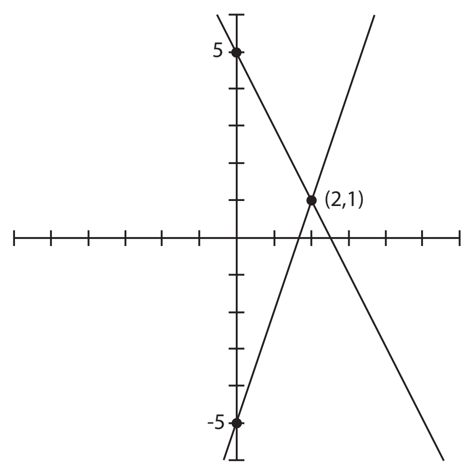
$$\begin{cases} 2x + y = 5 \\ 9x - 3y = 15 \end{cases}$$

Notice that

$$\begin{cases} 2x + y = 5 \\ 9x - 3y = 15 \end{cases}$$

is **equivalent** to $\begin{cases} y = -2x + 5 \\ y = 3x - 5 \end{cases}$.

We can graph the two lines indicated in the second system and find solutions to the system by looking for common points. In this case, because the slopes are different there will be only ONE solution, (2, 1)



9. *Solve this system symbolically.*

$$\begin{cases} 2x + y = 5 \\ 9x - 3y = 15 \end{cases}$$

One way to solve this symbolically is to rewrite *both* equations as equations that start y = (or x =).

$$\begin{cases} 2x + y = 5 \\ 9x - 3y = 15 \end{cases}$$

is **equivalent** to $\begin{cases} y = -2x + 5 \\ y = 3x - 5 \end{cases}$.

Thus,
-2x + 5 = 3x – 5. (Both sides are equal to y)

the linear graphs. Thus it is important to check estimated solutions in the original equations. If the equations are in **standard form** then they must be rewritten **in slope-intercept** form before graphing, either by hand or using a calculator. • **Symbolic Solution Strategies** fall into 3 types, and students learn to appraise the symbolic representations to determine which strategy will be most convenient. ✓ **Equivalent Form**: When the equations in a system are given in *ax* + *by* = *c* form, they can always be changed to the equivalent *y* = *ax* + *b* form. When the arithmetic is easy, this is a good strategy. ✓ **Substitution** —A second method of solving linear systems is useful when one of the given equations can be easily re-written in a form showing one variable as a function of the other, either y = some expression in x, or x = some expression in y. ✓ **Linear Combination** — A third method of solving linear systems relies on two basic principles:	Now, using Properties of Equality to solve this equation in one variable, 10 = 5x, 2 = x. We still need to find the y-value that corresponds to x = 2. We can do this by substituting x = 2 in either of the original equations. 2(2) + y = 5, so y = 1. Therefore, the solution is (2, 1). 10. *Solve this system by* ***substitution.*** $\begin{cases} 2x + y = 5 \\ 9x - 3y = 15 \end{cases}$ Another way to solve this system is to rewrite *only one* of the equations in y = or x = format. Since 2x + y = 5 has a term with coefficient 1 it is easy to rewrite as y = 5 – 2x. Now we **substitute** the expression 5 – 2x for y in the second equation. 9x – 3(**5 – 2x**) = 15. 9 x – 15 + 6x = 15. 15x = 30. X = 2. As before we need to find the corresponding y-value to complete the solution. 2(2) + y = 5, so y = 1. The solution is (2, 1). *11. Solve by using* ***linear combinations.*** 9x – 3y = 15 *and* 6x – 4y = 2. (Note: We indicate that the two equations are a system, for which common solutions are to be found, by bracketing the equations together, or by using "and" to link the equations.) 9x – 3y = 15 and 6x – 4y = 2 is equivalent to 18x – 6y = 30 and 18x – 12y = 6. Notice that this new system has been produced by *multiplying every term in the first equation by 2, and by multiplying every term in the second equation by 3.* If we graphed the first system we would see two lines. If we

(1) The solutions to any linear equation $ax + by = c$ are identical to the solutions of the equation $kax + kby = kc$. That is, multiplying both sides of a linear equation by the same (non-zero) number does not change the set of solutions.

(2) The solution of any system of linear equations is unchanged if one of the equations is replaced by a new equation formed by adding (or subtracting) the two original equations.

graphed the second system we would see the *same* two lines. However, the second system has "18x" in both equations. Now, *subtracting* $18x - 12y$ from $18x - 6y$ and *subtracting* 6 from 30, we have a new equation,
$6y = 24$. This has only one variable (because $18x - 18x = 0$), so we can solve, to get $y = 4$.
As usual we need to find the corresponding x-value. $9x - 3(4) = 15$. So $9x = 27$. So $x = 3$.
The solution is (3, 4).

12. Solve by using ***linear combinations.***
$9x - 2y = 15$ and
$6x + 4y = 2$.

We could again rewrite these equations so that they both have "18x" and subtract. Or we can focus our attention on the "y terms" and try to eliminate them.

$9x - 2y = 15$ and
$6x + 4y = 2$
is equivalent to
$18x - 4y = 30$ and
$6x + 4y = 2$
Notice that by multiplying only the first equation by 2 we have "4y" in both equations. Actually we have "-4y" in one equation and "4y" in the other. *Adding* the equations in the new system we have $24x = 32$. This has only one variable (because $-4y + 4y = 0$) so we can solve, to get $x = 4/3$.
As before we need the corresponding value of y to complete the solution.
$9(4/3) - 2y = 15$, so
$12 - 2y = 15$, so
$-2y = 3$, so
$y = -3/2$.

13. Solve the system
$9x - 3y = 15$ and
$6x - 2y = 2$.

If we are not directed to use a particular strategy then we can approach this system in whichever way we prefer. Say we decide to solve both of these equations for y. Then
$9x - 3y = 15$ is **equivalent** to
$-3y = 15 - 9x$ or $y = -5 + 3x$.

	Meanwhile 6x – 2y = 2 is **equivalent** to -2y = 2 – 6x, or y = -1 + 3x. The new system is y = -5 + 3x and y = -1 + 3x. Therefore, -5 + 3x = -1 + 3x. Using Properties of Equality on this equation we have -5 = -1. Now, this is never true, no matter what value x takes. Thus we have *NO solution for this system.* (Notice, had we chosen to graph these we would have noticed that both slopes are 3. The lines are parallel. There will be no common points.) *14. Solve the system* *9x – 3y = 15 and* *6x – 2y = 10.* Suppose we decide to use the **Linear Combinations** strategy. Then we might focus on making both equations have the same coefficient for y. We can make both y-terms have a coefficient of -6. An equivalent system is 18x – 6y = 30 and 18x – 6y = 30. Subtracting we have 0 = 0. This is true no matter what value x takes. Thus, there is an *infinite number of solutions for this system*. (Notice, had we chosen to graph we would have seen that 9x – 3y = 15 is exactly the same line as 6x – 2y = 10. All their points are common solutions.) Note: Every system of linear equations in 2 variables will have either 1 solution, no solution or an infinite number of solutions (corresponding to 2 lines with1 point of intersection, 2 parallel lines with no points of intersection, or 2 identical lines).

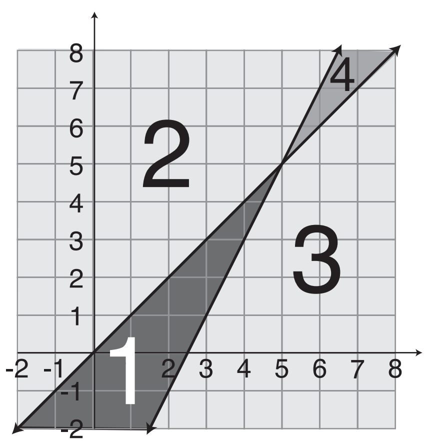
Solving Systems of Linear Inequalities The solution to a system of linear *equations* is the intersection of two lines (if they intersect). The solution to a system of distinct, non-disjoint linear *inequalities* is the intersection of two half-planes, which contains infinitely many points.	*15. Solve this system of linear inequalities.* 3x + y > 6 and 4x – 2y < 10. The original system is equivalent to Y > 6 – 3x and y > -5 + 2x.

planes, which contains infinitely many points.

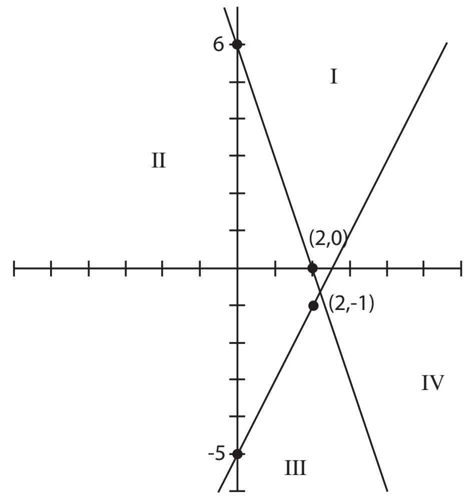
In general, there are four regions suggested by a system of linear inequalities such as

$$\begin{cases} y < x \\ y > 2x - 5 \end{cases}$$

These regions are numbered 1—4 in the diagram below. Region 1 contains the solutions to the system, for all of the points satisfy both inequalities. Regions 2 and 3 contain points that satisfy one or the other, but not both of the inequalities. These points are *not* solutions to *the system*, although each point is a solution to one of the inequalities. Region 4 contains points that satisfy neither of the equations. These points are also *not* solutions to the *system*.



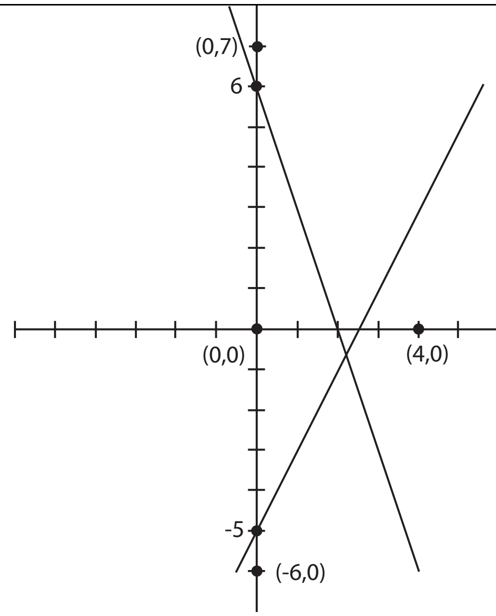
These inequalities can each be graphed by first graphing the boundary lines
Y = 6 – 3x and y = -5 + 2x.
This produces 4 regions of the plane.

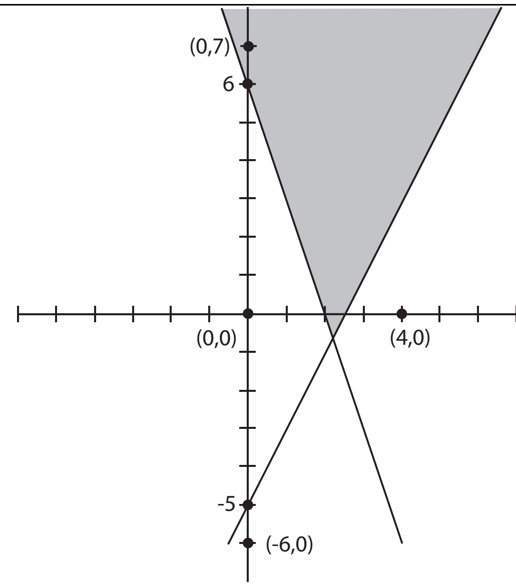


We can determine which of these regions contains the solutions by either

- Choosing points which are clearly in each of the regions , and checking these points in the original system of inequalities, to see which point is a solution, and therefore which region contains solutions, or
- Reasoning that we want points which are above (>) the boundary y =6 – 3x and also above (>) the boundary y = -5 + 2x.

In the case of the above graph we might choose points (0, 7), (0. 0), (-6, 0) and (4, 0) to represent the 4 regions. It is not necessary to choose points on the axes, but these make easy points to work with.

	 Checking (0, 7) we have 3(0) + 7 > 6 which is true, and 4(0) – 3(7) < 10 which is also true. (0, 7) is in the region which contains the solutions for the system. (Notice BOTH inequalities have to be checked.) Suppose we had checked (4, 0). Then we would have 3(4) + (0) > 6 which is true, and 4(4) – 2(0) < 10 which is not true. So (4, 0) is NOT a solution for both inequalities. The region that contains (4, 0) is not the desired solution set. We show the solution set as a shaded region.



(0,7)
6
(0,0)
(4,0)
-5
(-6,0)