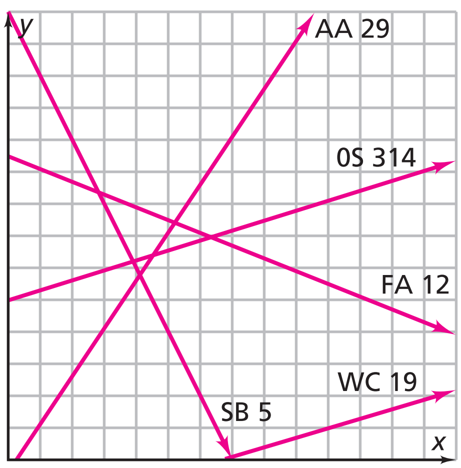
Investigation 4

Solving Systems of Linear Equations Symbolically

Every day of the year, thousands of airline flights crisscross the United States to connect large and small cities. Each flight follows a plan filed with air traffic control before take-off.

The flight paths of airplanes are not straight lines from take-off to landing. But parts of those paths are generally straight-line segments. At any hour of the day, the pattern of flight plans might look like the diagram below.



The diagram seems to show many potential mid-air collisions.

Why do you think such disasters almost never occur?

What information about flight paths is not shown on the diagram?

If planned flight paths are represented by equations, an air-traffic control system can calculate intersection points and warn of possible collisions. The equations for the flights in the diagram are shown in the table.

In this problem, you will explore a simplified air-traffic control system. You will ignore the height above the ground and time and consider only whether the flight paths of two planes intersect. These intersection points tell you which parts of the flight paths controllers need to examine more carefully to prevent collisions.

4.1 The $y = mx + b$ Case

The equations for the flight paths can be used to calculate the nine intersection points shown on the graph.

Getting Ready for Problem 4.1

A table of equations for the flight paths is at the right. To find the intersection of WC 19 and AA 29, you need to find the (x, y) pair that satisfies the system of linear equations below. (The bracket is a special notation used to indicate a system of equations.)

$$\begin{cases} y = 0.3x - 2 \\ y = 1.5x - 0.4 \end{cases}$$

Jeff writes the following to solve this system.

$$0.3x - 2 = 1.5x - 0.4$$
$$-1.2x = 1.6$$
$$x = \frac{1.6}{-1.2}$$
$$x = -\frac{4}{3}$$

Flight Paths

Airline/Flight	Equation
Apex Airlines Flight *AA 29*	$y = 1.5x - 0.4$
We-Care Air Flight *WC 19*	$y = 0.3x - 2$
Open Sky Airlines Flight *OS 314*	$y = 0.3x + 5$
Fly Away Airlines Flight *FA 12*	$y = -0.4x + 9.5$
Sky Bus Airlines Flight *SB 5*	$y = -2x + 14$

- Explain Jeff's reasoning.
- What does $x = -\frac{4}{3}$ tell you?
- How can you find the y-coordinate of the intersection point?

Problem 4.1 The $y = mx + b$ Case

A. Write and solve systems to find the intersections of these flight plans.

1. WC 19 and SB 5
2. SB 5 and AA 29
3. SB 5 and FA 12
4. FA 12 and AA 29

B. Study the work you did in Question A. Describe a strategy for solving any system of this form shown below.

$$\begin{cases} y = ax + b \\ y = cx + d \end{cases}$$

C. What could an air-traffic controller do if two flight plans intersect?

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4.2 The $ax + by = c$ Case

When a system of linear equations is in $y = mx + b$ form, it is easy to write a single linear equation with the same solution as the system.

$$\begin{cases} y = ax + b \\ y = cx + d \end{cases} \text{ becomes } ax + b = cx + d.$$

The equations in a linear system are not always given in $y = mx + b$ form. In this problem, you'll consider systems of this form:

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

Getting Ready for Problem 4.2

Suppose the solution to a situation requires you to find values of x and y that satisfy the system:

$$\begin{cases} 3x - y = 30 \\ x + y = 14 \end{cases}$$

One useful strategy for solving a linear system is to rewrite the equations in familiar equivalent forms.

- Write each equation in $y = mx + b$ form. Then, find a solution using the method you learned in Problem 4.1.
- Write each equation in $x = ky + c$ form. Then, find a solution. Are the solutions the same?
- Why might you expect both methods to give the same solution?

Problem 4.2 Solving Systems by Writing Equivalent Forms

A. Decide whether it is easier to write each equation in equivalent $y = mx + b$ form or equivalent $x = ky + c$ form. Then, write each equation in the form you chose.

1. $x + y = 3$

2. $x - y = -5$

3. $2x + y = -1$

4. $x - 2y = 8$

5. $9x + 6y = 12$

6. $-x + 4y = 10$

7. In parts (1)–(6), how did you decide which form to use?

B. Solve each system by writing the equations in $y = mx + b$ or $x = ky + c$ form and then using the strategy from Problem 4.1.

1. $\begin{cases} x + y = 3 \\ x - y = -5 \end{cases}$

2. $\begin{cases} 3x - y = 30 \\ x + y = 14 \end{cases}$

3. $\begin{cases} x + 6y = 15 \\ -x + 4y = 5 \end{cases}$

4. $\begin{cases} x - y = -5 \\ -2x + 2y = 10 \end{cases}$

C. Look back over your work from Question B.

1. What do you notice about the systems that makes this method a good one to use?

2. Describe the steps needed in using this method to solve a system.

D. **1.** What does it mean for two equations to be equivalent?

2. What does it mean to solve a linear system?

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4.3 Solving Systems by Substitution

Writing the equations in a linear system in $y = mx + b$ or $x = ky + c$ form is not always easy. Sometimes the arithmetic becomes messy.

For example, consider how you would solve this system.

$$\begin{cases} 3x - y = 5 \\ 2x + 5y = -8 \end{cases}$$

Solve both equations for y to get $y = mx + b$ form.

$$3x - y = 5 \qquad\qquad 2x + 5y = -8$$
$$-y = 5 - 3x \qquad\qquad 5y = -8 - 2x$$
$$y = 3x - 5 \qquad\qquad y = -\frac{8}{5} - \frac{2}{5}x$$

Set the right sides of the equations equal.

$$3x - 5 = -\frac{8}{5} - \frac{2}{5}x$$
$$15x - 25 = -8 - 2x$$
$$17x = 17$$
$$x = 1$$

In this problem, you'll look at another solution method that is easier in many cases.

Getting Ready for Problem 4.3

Check the reasoning in this method of solving a system of linear equations by *substitution* and see if you can explain why it works.

1. The system $\begin{cases} 3x - y = 5 \\ 2x + 5y = -8 \end{cases}$ is equivalent to the system $\begin{cases} y = 3x - 5 \\ 2x + 5y = -8. \end{cases}$

2. From that fact, any solution should satisfy $2x + 5(3x - 5) = -8$. Why is this equation an advantage over the two-equation system?

3. Solving this single equation for x, you get:

$$2x + 15x - 25 = -8$$
$$17x = 17$$
$$x = 1$$

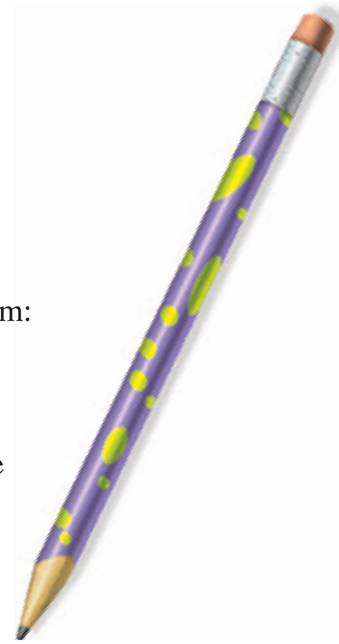
4. Then $y = 3(1) - 5 = -2$.

5. The ordered pair $(1, -2)$ satisfies both equations in the original system:

$3(1) - (-2) = 5$ $\qquad$ $2(1) + 5(-2) = -8$

So $(1, -2)$ is the solution.

- Does this strategy produce the *only* solution for both equations in the original system? Why?
- Which solution strategy do you think is easier for this system, writing the equations in $y = ax + b$ form and setting them equal or using substitution? Why?



Problem 4.3 Solving Systems by Substitution

A. Use substitution to solve each system.

1. $\begin{cases} 2x + y = -1 \\ x - 2y = 12 \end{cases}$ **2.** $\begin{cases} 4x + 2y = 6 \\ -3x - 7y = 1 \end{cases}$ **3.** $\begin{cases} x - y = -5 \\ -x + 4y = 10 \end{cases}$

4. $\begin{cases} 3x + y = 4 \\ 6x + 2y = 7 \end{cases}$ **5.** $\begin{cases} 3x + 2y = 10 \\ -6x - 4y = -20 \end{cases}$ **6.** $\begin{cases} x + y = 13 \\ x - y = 2 \end{cases}$

B. You may have been puzzled by the solution to two of the systems in Question A. Complete parts (1) and (2) for each of these two systems.

1. Graph the two lines to see if you can make sense of the situation.

2. Write both equations in $y = mx + b$ form to see if this helps you understand the results.

C. 1. Decide whether writing in equivalent form or substituting would be easier for solving the system. Then, solve the system.

a. $\begin{cases} 4x + y = 6 \\ -3x + y = 1 \end{cases}$ **b.** $\begin{cases} 2x + y = 3 \\ -3x + 7y = 1 \end{cases}$

2. For each system, explain why you chose the solution method.

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4.4 Solving Systems by Combination

You have already developed some useful strategies for solving a simple linear equation like $3x + 5 = 10$. You know that you can add or subtract the same quantity on both sides and preserve equality.

The same is true for multiplication or division. These ideas, called the *Properties of Equality,* can help you develop another method for solving linear equations. This method involves combining separate linear equations into one equation with only one variable.

Getting Ready for Problem 4.4

These steps show the *combination method* for solving $\begin{cases} x - y = 4 \\ x + y = 5. \end{cases}$

If $x - y = 4$ and $x + y = 5$, then

$$(x - y) + (x + y) = 4 + 5 \quad (1)$$
$$2x = 9 \quad (2)$$
$$x = 4.5 \quad (3)$$
$$x + y = 5 \quad (4)$$
$$4.5 + y = 5 \quad (5)$$
$$y = 0.5 \quad (6)$$

- Give reasons for steps 1–5.
- Why is adding the two original equations an advantage?
- Would subtracting the two original equations work just as well?

Problem 4.4 Solving Systems by Combination

A. Use the combination method to solve these linear systems.

1. $\begin{cases} -x + 4y = 3 \\ x + 2y = 5 \end{cases}$ **2.** $\begin{cases} 2x + 3y = 4 \\ 5x + 3y = -8 \end{cases}$ **3.** $\begin{cases} 2x - 3y = 4 \\ 5x - 3y = 7 \end{cases}$

B. 1. Explain why System B is equivalent to System A.

System A	System B
$\begin{cases} 3x + 2y = 10 \\ 4x - y = 6 \end{cases}$	$\begin{cases} 3x + 2y = 10 \\ 8x - 2y = 12 \end{cases}$

2. Rewriting System A as System B is a possible first step in solving the system by the combination method. Complete this solution process by combining the two equations in System B.

C. 1. Add the two equations in System A. Graph both equations in System A and the new equation you made by adding. What do the three equations have in common?

2. Graph System B and the new equation you made by adding. What do the three equations have in common?

3. Why does the graph you made with System B and the new equation help to solve the system?

D. In parts (1) and (2), write an equivalent system that is easy to solve by combining equations. Then find the solution. Check your work by solving the system with a different method.

1. $\begin{cases} 2x + 2y = 5 \\ 3x - 6y = 12 \end{cases}$ **2.** $\begin{cases} x + 3y = 4 \\ 3x + 4y = 2 \end{cases}$

E. 1. Decide whether equivalent form, substitution, or combination would be easiest for solving the system. Then, solve the system.

a. $\begin{cases} 2x + y = 5 \\ 3x - y = 15 \end{cases}$ **b.** $\begin{cases} x + 2y = 5 \\ x - 6y = 11 \end{cases}$ **c.** $\begin{cases} 2x + 6y = 7 \\ 3x - 2y = 5 \end{cases}$

d. $\begin{cases} 2x + y = 5 \\ -4x - 2y = -10 \end{cases}$ **e.** $\begin{cases} x + 2y = 5 \\ 3x + 6y = 15 \end{cases}$

2. For each system in part (1), explain how you decided which solution method to use.

F. Two of the systems in Question E did not have single solutions. How could you have predicted this before you started to solve them?

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